Linear programming

Discuss: class policies, hw, need for Sage, outline of the class, books, graph theory

Optimization problem

$$(P)\begin{cases}\text{minimize} & f(\mathbf{x})\\ \text{s.t.} & g_1(\mathbf{x}) \le b_1\\ & \vdots \\ & g_m(\mathbf{x}) \le b_m,\end{cases}$$

where $\mathbf{x} \in \mathbb{R}^n$, $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $b_i \in \mathbb{R}$. Program (P) is *linear* if f, g_i are linear functions. Reformulation:

$$(LP)\begin{cases}\text{minimize} & \mathbf{c}^T\mathbf{x}\\ \text{s.t.} & A\mathbf{x} \le \mathbf{b},\end{cases}$$

where $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$. Also maximize, $\ge$, $=$. Program (LP) is efficiently solvable (P-time). Note that $<$ or $>$ are NOT allowed.

History note: 1939 by Kantorovich[1], Dantzig (simplex method).

Examples of linear programming:

1: Diet problem: Formulate as a linear programming problem the following question: *How many apricots (x_1), bananas (x_2) and cucumbers (x_3) does one have to eat to get enough of vitamins A, B, and C while minimizing the cost?*

Need to know: % of daily value and cost:

	A	C	K	$	weight
apricots	60	26	6	1.53	155g
bananas	3	33	1	0.37	225g
cucumbers	2	7	12	0.18	133g

Solution:

$$(LP)\begin{cases}\text{minimize} & 1.53x_1 + 0.37x_2 + 0.18x_3\\ \text{s.t.} & 60x_1 + 3x_2 + 2x_3 \ge 100\\ & 26x_1 + 33x_2 + 7x_3 \ge 100\\ & 6x_1 + 1x_2 + 12x_3 \ge 100\end{cases}$$

Solution: $(x_1, x_2, x_3) = (1.4, 0.3, 7.6),$. The answer is $1.4 \cdot 155g$, $0.3 \cdot 225g$, and $7.6 \cdot 133g$ of apricots, bananas, and cucumbers respectively and the cost is \$3.62.

HW: Feed the professor!

[1] Full professor at age 22.

2: Farming: A farmer has 12 acres of land to plant either soybeans or corn. At least 7 acres have to be planted. Planting one acre of soybeans costs \$200 and one acre of corn costs \$100. Budget for planting is \$1500. The sale from one acre of soybeans is \$500 and from corn is \$300. How many acres of what should be planted to maximize profit?

Linear programming was the biggest invention in mathematics in the last century - if measured by \$.

Solution:

$$(LP)\begin{cases} \text{minimize} & (500-200)soy + (300-100)corn \\ \text{s.t.} & 200soy + 100corn \le 1500 \\ & soy + corn \le 12 \\ & soy + corn \ge 7 \\ & soy \ge 0 \\ & corn \ge 0 \end{cases}$$

Solution is 3 acres of soy and 9 acres of corn. Profit is \$2700.

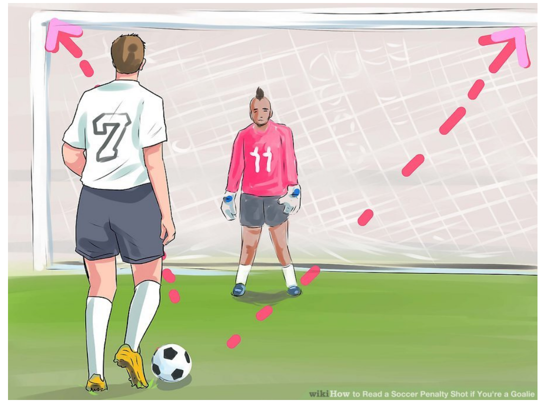
APMonitor and Sage writeup of the problem.

```
Model farmer  ! APmonitor
  Variables
    soy  = 0, >= 0
    corn = 0, >= 0
  End Variables
  Equations
  maximize 500*soy + 300*corn  - 200*soy - 100*corn
    200*soy + 100*corn <= 1500
    soy + corn <= 12
    soy+corn >= 7
  End Equations
End Model
```

```
p = MixedIntegerLinearProgram(maximization=True)  # Sage
x = p.new_variable(nonnegative=True)
p.set_objective( 500*x[0] + 300*x[1] - 200*x[0] - 100*x[1])
p.add_constraint( 200*x[0] + 100*x[1] <= 1500 )
p.add_constraint( x[0] + x[1] <= 12 )
p.add_constraint( x[0] + x[1] >= 7 )
print "Profit $", p.solve()
print "Soybeans",p.get_values(x[0]),"acres, Corn",p.get_values(x[1]),"acres"
```

3: 2-Player Zero-Sum Games:

In penalty kicks in soccer (football in World\USA), the kicker (number 7) kicks the ball and usually tries to aim at one of the top corners. The goalie (number 11) tries to guess which corner the kicker kicks and jumps towards one of the corner. If the goalie has a correct guess, there is a very good change for the goalie to catch the ball. If the goalie guesses wrong, it is a goal unless the kicker messes up.



Assume you are the kicker and you know that the goalie has a handicap that if you shoot to the left and the goalie jumps left, there is only 10% chance for you to score but if you kick to the right and the goalie jumps to the right, there is 50% chance of scoring. If the goalie jumps in the opposite direction than your kick, you have 95% chance of scoring. Should you kick the ball to the left or to the right?

If you always kick to the right, the goalie will always jump to the right and you score 0.5 goals per kick. It is better to pick left or right with some probability. What is the best left-right probability subject to the goalie picking his random jumps to counter your strategy as much as possible?

Solution: Lets create a scoring table. In the row, the kicker picks left or right, then goalie picks left or right (not knowing the kicker's pick) and the outcome is in the table.

goalie

kicker

	left	right
left	0.1	0.95
right	0.95	0.5

To formulate this as a linear program, we start with variables ℓ and r. We also add a variable s, which is the expected score (number of goals).

$$(LP)\begin{cases} \text{maximize} & s \\ \text{s.t.} & 0.1\ell + 0.95r \geq s \\ & 0.95\ell + 0.5r \geq s \\ & \ell + r = 1 \\ & \ell \geq 0 \\ & r \geq 0 \end{cases}$$

The solution is approximately $\ell = 0.346$, $r = 0.654$ and $s = 0.6557$. Notice that this randomized strategy gives at least 0.65557 no matter what is the strategy of the goalie.

4: Ropes: We are producing packages of two 15cm ropes and one 20cm rope (say for some kid's game). Suppose we have have 400 times 50cm-rope and 100 times 65cm-ropes. How should we cut the ropes to maximize the number of produced packages?

Solution: #15 cm = A, #20 cm = B,

$$\begin{aligned}50cm &= 15+15+20\ldots x_1\ldots 2A+B\\ &= 20+20\ldots x_2\ldots 2B\\ &= 15+15+15\ldots x_3\ldots 3A\end{aligned}$$

$$\begin{aligned}65cm &= 20+20+20\ldots y_1\ldots 3B\\ &= 15+15+15+15\ldots y_2\ldots 4A\\ &= 20+15+15+15\ldots y_3\ldots B+3A\\ &= 20+20+15\ldots y_4\ldots 2B+A\end{aligned}$$

$$(LP)\begin{cases}\text{maximize} & p\\ \text{s.t.} & p\le \frac{1}{2}A\\ & p\le B\\ & A=2x_1+3x_3+4y_2+3y_3+y_4\\ & B=x_1+2x_2+3y_1+y_3+2y_4\\ & 400\ge x_1+x_2+x_3\\ & 100\ge y_1+y_2+y_3\end{cases}$$

Solution:

$$p=528.5, x_1=400, x_2=0, x_3=0, y_1=14.28, y_2=0, y_3=85.71, y_4=0$$

We are missing that x_i, y_j are actually integers! Adding the constraint that the variables are integers result in significantly more difficult problem.